ON THE MEAN CONVERGENCE FOR DOUBLE ARRAYS OF PAIRWISE INDEPENDENT RANDOM VECTORS IN HILBERT SPACES

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Abstract: In this paper, we prove a theorem on convergence in mean of order p for double arrays of pairwise independent random vectors in Hilbert Space, where $1 \le p < 2$. The main result extends Theorem 2.1 of Bao et al. [2] and Theorem 2.1 of Thanh [8]. The proof is based on the von Bahr–Essen inequality for pairwise independent random vectors taking values in Hilbert spaces.

Keywords: Double array; convergence in mean; Hilbert space; pairwise independence; random vector; uniform integrability.

1 Introduction and preliminaries

Let $1 \le p < 2$, the current paper obtains mean convergence theorems for double arrays of pairwise independent random vectors in Hilbert spaces that satisfy certain uniformly integrable conditions. A similar result for pairwise independent real-valued random variables was obtained in [2]. To obtain the result in the Hilbert spaces setting, we use the von Bahr–Esseen inequality for pairwise independent random vectors which was proved recently in [1].

Mean convergence theorems and laws of large numbers for double arrays of independent and dependent random variables are studied by several authors. We refer the reader to [2, 4, 5, 6, 8] and the references therein.

Now, we will recall the notion of uniform integration. A sequence of random variables $\{X_n, n \ge 1\}$ is said to be *uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\ge1} \mathbb{E}(|X_n|\mathbf{1}(|X_n| > a)) = 0.$$

Following Thanh [9], we introduce the concept of uniformly integrable for a double array of random variables as follows. A double array of random variables $\{X_{mn}, m \ge 1, n \ge 1\}$ is said to be uniformly integrable if

$$\lim_{a\to\infty} \sup_{m\ge1,n\ge1} \mathbb{E}(|X_{mn}|\mathbf{1}(|X_{mn}| > a)) = 0.$$

Throughout this paper, $\mathcal{H}$ denotes a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, the corresponding norm $\|\cdot\|$, and an orthonormal basis $\{e_j, j \in B\}$. The *expected*

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value or *mean* of a random vector X, denoted by $\mathbb{E}(X)$, is defined to be the *Pettis integral* provided it exists. That is, X has expected value $\mathbb{E}(X) \in \mathcal{H}$ if $f(\mathbb{E}(X)) = \mathbb{E}(f(X))$ for every $f \in \mathcal{H}^*$, $\mathcal{H}^*$ denotes the *dual* space of all continuous linear functions on $\mathcal{H}$. If $\mathbb{E}(\|X\|) < \infty$, then (see, e.g., [7, p. 40]) X has an expected value. But the expected value can exist when $\mathbb{E}(\|X\|) = \infty$; for an example, see [7, p. 41].

The following lemma is the von Bahr–Esseen inequality for pairwise independent random vectors taking values in Hilbert spaces. It was proved recently in [1]. The von Bahr–Esseen inequality for pairwise independent real-valued random variables was proved by Chen et al. [3].

Lemma 1.1. *Let $1 \le p \le 2$ and let $\{X_1, X_2, \ldots, X_n\}$ be a collection of n pairwise independent mean zero random vectors in $\mathcal{H}$ with $\mathbb{E}(\|X_k\|^p) < \infty$ for all $1 \le k \le n$. Then*

$$\mathbb{E}\left\|\sum_{k=1}^{n} X_k\right\|^p \le C_p \sum_{k=1}^{n} \mathbb{E}\|X_k\|^p, \tag{1.1}$$

where C_p is a constant depending only on p.

2 Main results

In this section, we prove a theorem on convergence in mean of order p for double arrays of pairwise independent random vectors in Hilbert spaces, where $1 \le p < 2$. The following theorem is the main result of this paper. It extends the main result of Bao et al. [2], and therefore extends the main result of Thanh [8] to the case of the Hilbert space-valued random vectors.

Theorem 2.1. *Let $1 \le p < 2$ and let $\{X_{mn}, m \ge 1, n \ge 1\}$ be a double array of pairwise independent random vectors in $\mathcal{H}$ such that $\{\|X_{mn}\|^p, m \ge 1, n \ge 1\}$ is uniformly integrable. Then*

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - \mathbb{E}(X_{ij}))}{(mn)^{1/p}} \overset{\mathcal{L}_p}{\to} 0 \text{ as } m \vee n \to \infty. \tag{2.1}$$

Proof. Since the array $\{\|X_{mn}\|^p, m \ge 1, n \ge 1\}$ is uniformly integrable, we have

$$\lim_{a\to\infty} \sup_{m\ge1,n\ge1} \mathbb{E}\left(\|X_{mn}\|^p \mathbf{1}(\|X_{mn}\| > a)\right) = 0. \tag{2.2}$$

Let $\varepsilon > 0$ be arbitrary. It follows from (2.2) that there exists $M > 0$ such that

$$\mathbb{E}(\|X_{mn}\|^p \mathbf{1}(\|X_{mn}\| > M)) < \varepsilon \text{ for all } m \ge 1, n \ge 1. \tag{2.3}$$

For $m \ge 1, n \ge 1$, set

$$X'_{mn} = X_{mn}\mathbf{1}(\|X_{mn}\| \le M),\ X''_{mn} = X_{mn}\mathbf{1}(\|X_{mn}\| > M). \tag{2.4}$$

Then, for all $m \ge 1, n \ge 1$, we have

$$\mathbb{E}\left\|X''_{mn} - \mathbb{E}X''_{mn}\right\|^p \le 4\mathbb{E}\left\|X''_{mn}\right\|^p = 4\mathbb{E}(\|X_{mn}\|^p \mathbf{1}(\|X_{mn}\| > M)) < 4\varepsilon. \tag{2.5}$$

To prove (2.1), we need to show that

$$\lim_{m\vee n\to\infty} \frac{\mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))\right\|^p}{mn} = 0. \tag{2.6}$$

We have

$$\begin{aligned} &\mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))\right\|^p \\ &\le 2^{p-1}\left(\mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij}-\mathbb{E}(X'_{ij}))\right\|^p + \mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X''_{ij}-\mathbb{E}(X''_{ij}))\right\|^p\right). \end{aligned} \tag{2.7}$$

We will now estimate the two terms on the right-hand side of (2.7) as follows.

Firstly, since $\{X'_{ij}-\mathbb{E}X'_{ij}, i\ge 1, j\ge 1\}$ is a double array of pairwise independent random vectors in $\mathcal{H}$, we have

$$\begin{aligned} \mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij}-\mathbb{E}(X'_{ij}))\right\|^p &\le \left(\mathbb{E}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij}-\mathbb{E}(X'_{ij}))\right)^2\right)^{p/2} \\ &= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\left(X'_{ij}-\mathbb{E}(X'_{ij})\right)^2\right)^{p/2} \\ &\le \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\left(X'_{ij}\right)^2\right)^{p/2} \le (mnM^2)^{p/2}. \end{aligned} \tag{2.8}$$

where we have applied Liapunov's inequality and the assumption $1\le p<2$ in the first inequality, and (2.4) in the last inequality.

Secondly, since $\{X''_{ij}-\mathbb{E}X''_{ij}), i\ge 1, j\ge 1\}$ is a double array of pairwise independent mean zero random vectors in $\mathcal{H}$, we have

$$\begin{aligned} \mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X''_{ij}-\mathbb{E}(X''_{ij}))\right\|^p &\le C_p\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\left\|X''_{ij}-\mathbb{E}X''_{ij}\right\|^p \\ &\le 4C_p mn\varepsilon. \end{aligned} \tag{2.9}$$

where we have applied Lemma 1.1 in the first inequality, and (2.5) in the last inequality.

It follows from (2.7)–(2.9), that

$$\mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))\right\|^p \le 2^{p-1}\left((mnM^2)^{p/2}+4C_p mn\varepsilon\right). \tag{2.10}$$

Therefore, we can write

$$\frac{\mathbb{E}\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))\right\|^p}{mn} \leq 2^{p-1}\left(\frac{M^p}{(mn)^{(2-p)/2}}+4C_p\varepsilon\right). \tag{2.11}$$

For $p<2$, C_p is a constant depending only on p and $\varepsilon>0$ is arbitrary, the conclusion (2.6) follows from (2.11). □

Remark 2.2. If $\{X_{mn}, m\geq 1, n\geq 1\}$ is a double array and stochastically dominated by a random variable X and $\mathbb{E}(|X|^p\log^+(|X|^p))<\infty$, then it is easy to show that $\{\|X_{mn}\|^p, m\geq 1, n\geq 1\}$ is uniformly integrable. Therefore, from Theorem 2.1, we can obtain a result of Hong and Hwang [4, Theorem 2.5].

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TÓM TẮT

VỀ SỰ HỘI TỤ THEO TRUNG BÌNH CỦA MẢNG KÉP CÁC VECTOR NGẪU NHIÊN ĐỘC LẬP ĐÔI MỘT NHẬN GIÁ TRỊ TRONG KHÔNG GIAN HILBERT

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Trong bài báo này, chúng tôi thiết lập được định lý hội tụ theo trung bình cấp p cho mảng hai chiều các vector ngẫu nhiên độc lập đôi một nhận giá trị trong không gian Hilbert, với $1 \leq p < 2$. Kết quả chính của chúng tôi mở rộng Định lí 2.1 của Bảo và cộng sự [2] và Định lý 2.1 của Thành [8]. Phép chứng minh dựa vào bất đẳng thức von Bahr–Essen cho các vector ngẫu nhiên độc lập nhận giá trị trong không gian Hilbert.

Từ khóa: Mảng hai chiều; sự hội tụ theo trung bình; không gian Hilbert; tính độc lập đôi một; vector ngẫu nhiên; tính khả tích đều.